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EFFECT OF THE EARTH'S ROTATION ON TRAJECTORIES OF
THE ANGLED ARROW PROJECTILE

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EFFECT OF THE EARTH'S ROTATION ON TRAJECTORIES OF THE ANGLED ARROW PROJECTILE

Prepared by:

Oscar T. Schultz

ABSTRACT: The effect of the earth's rotation on trajectories of the Angled Arrow Proje[illegible] obtained by three methods: a numerical [illegible]ation of the equations of motion including the Coriolis force and by two approximate integration methods applied to the linear differential equations which define the variations from the range table trajectories produced by the earth's rotation. A comparison of sample computations by the three methods indicates that the more exact method of correcting range table traject[illegible] sufficiently accurate for [illegible] purposes while the less exact method is sufficient to give the general magnitude of the effect of the earth's rotation but is probably inadequate for use in the AAP fire control computers.

U. S. NAVAL ORDNANCE LABORATORY
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While the existing range tables for the Angled Arrow Projectile are not regarded as final and their revision will necessitate corresponding changes in the fire control computers now being designed, it is desirable that the ballistic design of these computers be sufficiently complete that any modifications necessary to accomodate them to the final range tables will be of a minor nature. Since the effects of the earth's rotation are not included in the present range tables and their later inclusion might require more extensive alterations of the computers than will other improvements in the range [illegible], consideration should be given to the [illegible]mate ne[illegible] accounting for the earth's rotatio[illegible]

This report [illegible] methods for [illegible] practical co[illegible]tion of the effect of the earth's rotation, on the basis of which an estimate can be made of the error committed by neglecting this effect. These methods also suggest possible means of including an earth's rotation correction in the fire control computers.

W. G. SCHINDLER
Rear Admiral, USN
Commander

R. E. Hightower

R. E. HIGHTOWER
By direction

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EFFECT OF THE EARTH'S ROTATION ON TRAJECTORIES OF THE ANGLED ARROW PROJECTILE

INTRODUCTION

1. The design of the ballistic section of the Sperry Guidance Computer for the Angled Arrow Projectile is being based on the Preliminary Range Tables for the Angled Arrow Projectile contained in NAVORD Report 2160, and these tables are also being used in the conversion of a Ford Mk 1A Computer to serve as an AAP Launching Computer. As implied by their title, these tables are admittedly of a provisional nature and are subject to revision as a result of changes in the projectile configuration or the obtaining of more accurate drag information on the present design of projectile. The purpose of such tables is to allow the designs of the two computers to proceed before the definitive tables become available. The computers constructed with this ballistic information will be used in the proposed tests of the AAP System at the Chesapeake Bay Annex of NRL. In these tests no actual projectiles will be fired and the imaginary projectiles of the tests will be assumed to have a ballistic behavior which is exactly described by the present range tables. There is no doubt that this assumption is justified for the purposes of the CBA tests.

2. In ship board firings, of course, it is necessary that the range tables which are represented in the Guidance Computer accurately describe the behavior of the actual projectile. It is expected that this requirement will necessitate a revision of the present range tables, along with corresponding modifications of the computer, before the system is installed aboard ship; and it is hoped that both the design of the projectile and the experimental determination of the drag coefficient will be sufficiently definite by that time that the revisions can be made with some certainty of finality. Since the Guidance Computer will correct for the ballistic deficiencies of the Launching Computer, it is not so important that the Launching Computer be based on accurate range tables, and only the Guidance Computer is considered in the following discussion.

3. The present range tables are intended to be sufficiently accurate that only minor changes in the computer will be required to accommodate it to the final tables. Such changes might consist of changing the values of fixed resistances or the functions represented by function potentiometers, etc.,

but a modification requiring the addition of function potentiometers which were not originally present would probably be regarded as a major change, especially if the added potentiometers required mechanical inputs representing variables which were not present at all originally.

4. It seems likely that any changes in the computer which would be required to accommodate a refinement of the drag coefficient, or a change in the drag coefficient brought about by small changes in the projectile design, could be accomplished by the minor changes described above. There are certain other refinements which might also be accounted for in this way: the diminution of the gravitational force with altitude, the change in the direction of this force with the horizontal travel of the projectile, and changes in the density and temperature structures of the standard atmosphere.

5. There is, however, at least one deficiency in the present range tables the removal of which might require a major change in the computer, and that is the neglect of the earth's rotation. The earth's rotation brings about a departure of the projectile from the vertical plane containing the initial velocity vector, an effect of a kind which is entirely absent from the present tables except in the presence of a wind. Hence the only possibility of allowing for this effect with the present computer design is to introduce a fictitious wind which would have the same effect on the trajectory as the earth's rotation, but there is no reason to believe that this method would produce a sufficiently accurate correction. Furthermore, the deflection of the projectile caused by the earth's rotation is a function of the latitude, so that even if it were possible to represent the effect by introducing a fictitious wind, this wind would have to be varied as a function of latitude, a quantity which is not involved in the present ballistic solution.

6. Whether or not the neglect of the earth's rotation is justified depends on the magnitude of its effect compared with other errors in the range tables and also on the magnitude of all such ballistic errors relative to other errors in the AA System. In NAVORD Report 2160 an example is given in which the error due to this cause is about 60 ft at a time of flight of 19 sec. It seems quite possible that this error is negligible as compared with other errors in the present tables, but it is to be hoped that knowledge of the drag coefficient will improve to the extent that this error becomes more serious. Even if all errors in the drag coefficient could be eliminated, there might remain errors in accounting for wind, atmospheric density, etc. which would produce errors comparable to the effect of the earth's rotation. These errors, however, would appear as random

over a large number of engagements while the error due to neglect of the earth's rotation is systematic and therefore more serious.

7. A comparison of the relative importance of ballistic errors of all kinds with other errors in the AA System is made difficult by the great influence of target maneuver on the magnitude of the prediction error, and this question will not be considered here. It should be pointed out that the AAF reduction of prediction errors which is expected to be brought about by the use of high muzzle velocity and corrective deflection justifies a greater effort to reduce ballistic errors than would be the case for a conventional antiaircraft system.

THE EQUATIONS OF MOTION

8. Various degrees of refinement are possible in a treatment of the effects of the earth's rotation. In a rigorous treatment it would be desirable as a part of the problem to consider the variation of the gravitational force with the altitude and horizontal travel of the projectile and with the latitude of the gun. These effects are neglected in the following discussion because they have not been taken into account in the present preliminary range tables. If the computer design based on these range tables included also a correction for the earth's rotation based on the theory presented here, any subsequent refinements of the theory could almost certainly be accounted for by minor changes in the computer.

9. Let $\underline{r}_0$ be the position vector represented by the preliminary range tables from the gun to the projectile at time t after launching. Using a prime to denote the "apparent" derivative of a vector, i.e., the rate of change of the vector as seen by an observer who is fixed to the rotating earth, $\underline{r}_0''$ is taken as the acceleration of the projectile (regarded as a particle), $m\underline{r}_0''$ as the force acting on it (m being its mass), and this force is the resultant of the forces of gravity and air resistance. The force of gravity is taken as $-mg\underline{k}$ where $\underline{k}$ is a unit vector in the direction of the zenith at the gun and g is the scalar "acceleration due to gravity". The conventional direction of $\underline{k}$ and value of g include the centrifugal force due to the earth's rotation as well as the true gravitational force and in this sense the preliminary range tables do not entirely ignore the earth's rotation. This statement indicates the difficulty in treating the earth's rotation rigorously without at the same time introducing topics which seem superficially unrelated to it.

10. The force due to the air resistance is assumed to have a magnitude $1/2\,\rho v_0^2 AC$, where $v_0 = |\underline{r}_0'|$ is the speed of

the projectile through the air, ρ is the density of the air, A is the cross sectional area of the projectile, and C is the drag coefficient. The density of the air is that at the position of the projectile and hence $\rho = \rho(\underline{r}_0)$. The drag coefficient is an empirical function $C = C(M_0)$ of the Mach number $M_0 = v_0/a$, where $a = a(\underline{r}_0)$ is the speed of sound at the position of the projectile and depends on the temperature at that point. The air resistance acts in a direction opposite to the velocity of the projectile so that in magnitude and direction the air resistance force is

$$-1/2\,\rho v_0^2 AC \frac{\underline{r}_0'}{|\underline{r}_0'|} = -1/2\,\rho v_0 AC \underline{r}_0' .$$

The equation of motion of the projectile is then

$$m\underline{r}_0'' = -mg\underline{k} - 1/2\,\rho v_0 AC \underline{r}_0' . \qquad (1)$$

With $C(M_0)$ having the form determined experimentally and ρ and a being the functions of altitude of the NACA standard atmosphere, the present preliminary range tables represent the solutions of equation (1) for various initial conditions and non standard conditions, obtained by numerical integration.

11. Considering next the earth's rotation, let $\underline{r}$ be the position vector of the projectile when this influence is taken into account. As before, a prime will be used to indicate the "apparent" derivative of a vector while, in addition, a dot will be used to denote its true derivative. The total force acting on the projectile is now $m\ddot{\underline{r}}$ and, as before, is the sum of the gravitational and air resistance forces. The force of gravity is again $-mg\underline{k}$ while the air resistance force is now $-1/2\,\rho v AC\underline{r}'$, where $\underline{r}'$ is the velocity of the projectile relative to the air mass (assumed fixed to the rotating earth), $v = |\underline{r}'|$, and $C = C(M)$ with $M = v/a$. The equation of motion is then

$$m\ddot{\underline{r}} = -mg\underline{k} - 1/2\,\rho v AC\underline{r}' . \qquad (2)$$

12. Now if $\underline{F}$ is any vector, the relation between its true derivative and its "apparent" derivative in a frame of reference rotating with angular velocity $\underline{w}$ (here the angular velocity of the earth) is

$$\underline{\dot{H}} = \underline{H}' + (\underline{w} \times \underline{H}). \qquad (3)$$

Applying equation (3) with $\underline{H}$ replaced by $\underline{\dot{r}}$,

$$\underline{\ddot{r}} = (\underline{\dot{r}})' + (\underline{w} \times \underline{\dot{r}}).$$

The vector $\underline{\dot{r}}$ in the right member of (4) can be expressed by applying equation (3) again with, now, $\underline{H}$ replaced by $\underline{r}$, with the result

$$\underline{\ddot{r}} = [\underline{r}' + (\underline{w} \times \underline{r})]' + \{\underline{w} \times [\underline{r}' + (\underline{w} \times \underline{r})]\}$$

$$= \underline{r}'' + (\underline{w}' \times \underline{r}) + 2(\underline{w} \times \underline{r}') + [\underline{w} \times (\underline{w} \times \underline{r})] .$$

Since the angular velocity of the earth is constant, $\underline{w}' = 0$ and

$$\underline{\ddot{r}} = \underline{r}'' + 2(\underline{w} \times \underline{r}') + [\underline{w} \times (\underline{w} \times \underline{r})] . \qquad (5)$$

Now the magnitude of $\underline{w}$ is $w = 7.2921 \times 10^{-5}$ rad/sec and the third term on the right of (5) involves w^2 and is negligible as compared with the term involving w to the first power. Furthermore, no significance could be attached to this term without at the same time treating $\underline{k}$ and g in greater detail. Neglecting this term in (5) and substituting in (2), the latter equation becomes

$$m\underline{r}'' = - mg\underline{k} - 1/2\, \rho\, vAC\underline{r}' - 2m(\underline{w} \times \underline{r}') , \qquad (6)$$

which differs from (1) by the inclusion of the (fictitious) Coriolis Force $2m(\underline{w} \times \underline{r}')$. It would be possible to integrate equation (6) numerically, as has been done in a numerical example considered later, but the presence of the additional term would necessitate a complete integration for every latitude and gun bearing, in addition to the arguments of the present range tables, so that a prohibitive amount of work would be required. Furthermore, this procedure would make no use of the present range tables, nor would the results be in a form which would indicate a simple mechanization of the Computer.

13. These difficulties can be avoided by obtaining a differential equation for the difference between the vector $\underline{r}_0$ of the present range tables and the vector $\underline{r}$ of equation (6). Approximations can be made in this equation making it much easier to solve than is equation (6), and the solution can be carried out in such a way that each trajectory of the present range tables gives rise to the corrections to $\underline{r}_0$ which are necessary for all possible values of latitude and gun bearing, without the need of a separate numerical integration for each case. The method used is based on that of F. R. Moulton's <u>New Methods in Exterior Ballistics</u>.

14. Let $\underline{\Delta} = \underline{r} - \underline{r}_0$ be the vector which must be added to the position vector $\underline{r}_0$ of the present range tables to correct for the effect of the earth's rotation. Substituting (1) and (6) in $\underline{\Delta}'' = \underline{r}'' - \underline{r}_0''$,

$$\underline{\Delta}'' = -\frac{A}{2m}\left\{\rho(\underline{r})vC(M)\underline{r}' - \rho(\underline{r}_0)v_0C(M_0)\underline{r}_0'\right\} - 2(\underline{w} \times \underline{r}') ,$$

$$\underline{\Delta}'' = -\frac{A}{2m}\left\{\rho(\underline{r})vC(M)(\underline{r}_0' + \underline{\Delta}') - \rho(\underline{r}_0)v_0C(M_0)\underline{r}_0'\right\}$$
$$- 2\underline{w} \times (\underline{r}_0' + \underline{\Delta}'),$$

$$\underline{\Delta}'' = -\frac{A}{2m}\left\{\left[\rho(\underline{r})vC(M) - \rho(\underline{r}_0)v_0C(M_0)\right]\underline{r}_0' + \rho(\underline{r})vC(M)\underline{\Delta}'\right\}$$
$$- 2(\underline{w} \times \underline{r}_0') - 2(\underline{w} \times \underline{\Delta}') . \qquad (7)$$

It is convenient in the following steps to work with scalar equations instead of the vector equation (7). To obtain these equations, let $\underline{k}$, as before, be a unit vector in the direction of the zenith at the gun, and let $\underline{i}$ and $\underline{j}$ be unit vectors in the horizontal plane with $\underline{i}$ in the vertical plane containing the initial velocity of the projectile (the vertical plane of fire) and $\underline{j}$ perpendicular to this plane in such a sense that $\underline{i}$, $\underline{j}$, $\underline{k}$ form a right hand system. Then let

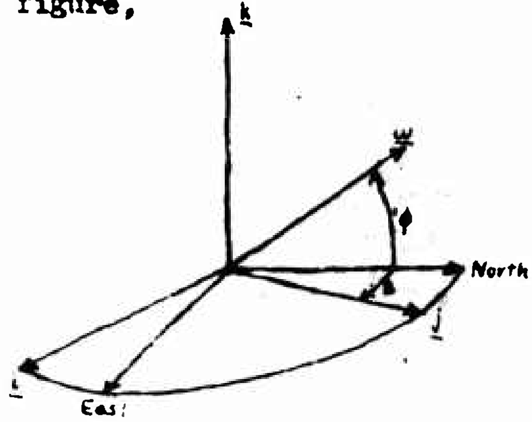
$$\underline{r} = x\,\underline{i} + y\,\underline{j} + z\,\underline{k} ,$$

$$\underline{r}_0 = x_0\underline{i} + y_0\underline{j} + z_0\underline{k} ,$$

$$\underline{\Delta} = \xi\underline{i} + \eta\underline{j} + \zeta\underline{k} .$$

In writing the "apparent" derivatives $\underline{r}'$, etc. the derivatives of x, etc. will be indicated by dots, there being no distinction between the true and apparent derivatives of a scalar.

15. Let ϕ be the latitude of the gun and B be the bearing angle of the gun, i.e., the angle between the vertical plane of fire and the vertical plane containing the North direction. The vector $\underline{w}$ representing the earth's angular velocity lies in the vertical plane containing the North direction and forms an angle ϕ with the horizontal, as shown in Fig. 1. From the figure,



$$\underline{w} = -w\cos\phi\sin B\,\underline{i} + w\cos\phi\cos B\,\underline{j} + w\sin\phi\,\underline{k},$$

Fig. 1

and the vector products appearing in (7) are

$$\begin{aligned}\underline{w} \times \underline{r}_0' = {} & (w\dot{z}_0\cos\phi\cos B - w\dot{y}_0\sin\phi)\underline{i} \\ & +(w\dot{x}_0\sin\phi + w\dot{z}_0\cos\phi\sin B)\underline{j} \\ & -(w\dot{y}_0\cos\phi\sin B + w\dot{x}_0\cos\phi\cos B)\underline{k}\,,\end{aligned}$$

$$\begin{aligned}\underline{w} \times \underline{\Delta}' = {} & (w\dot{\zeta}\cos\phi\cos B - w\dot{\eta}\sin\phi)\underline{i} \\ & +(w\dot{\xi}\sin\phi + w\dot{\zeta}\cos\phi\sin B)\underline{j} \\ & -(w\dot{\eta}\cos\phi\sin B + w\dot{\xi}\cos\phi\cos B)\underline{k}\,.\end{aligned}$$

16. In the standard atmosphere the density $\rho(\underline{r})$ and speed of sound $a(\underline{r})$ depend only on the altitude, that is, on the vertical component of $\underline{r}$, and will be denoted by $\rho(z)$ and $a(z)$. Finally, since the trajectories of the range tables lie in the vertical plane of fire and $\underline{i}$ is perpendicular to this plane, $x_0 = \dot{x}_0 = 0$. The scalar equations corresponding to (7) are then

$$\ddot{\xi} = -\frac{A}{2m}\rho(z)vC(M)\dot{\xi} - 2w\dot{z}_0 \cos\phi \cos B + 2w\dot{y}_0 \sin\phi$$
$$- 2w\dot{\zeta}\cos\phi\cos B + 2w\dot{\eta}\sin\phi ,$$

$$\ddot{\eta} = -\frac{A}{2m}\left\{\left[\rho(z)vC(M) - \rho(z_0)v_0C(M_0)\right]\dot{y}_0 + \rho(z)vC(M)\dot{\eta}\right\} \tag{8}$$
$$- 2w\dot{z}_0 \cos\phi \sin B - 2w\dot{\xi}\sin\phi - 2w\dot{\zeta}\cos\phi\sin B,$$

$$\ddot{\zeta} = -\frac{A}{2m}\left\{\left[\rho(z)vC(M) - \rho(z_0)v_0C(M_0)\right]\dot{z}_0 + \rho(z)vC(M)\dot{\zeta}\right\}$$
$$+ 2w\dot{y}_0 \cos\phi \sin B + 2w\dot{\eta}\cos\phi\sin B$$
$$+ 2w\dot{\xi}\cos\phi\cos B.$$

THE LINEAR EQUATIONS FOR THE DEFLECTION

17. Up to this point no approximations have been made and, in the presence of (1), equations (8) are equivalent to (6). The approximations which are first made in equations (8) are motivated by the desirability of obtaining linear differential equations for ξ, η, ζ and are justified by the assumption that these quantities and their derivatives are so small that their products and powers can be neglected. To make use of this assumption, consider the expression $\rho(z)vC(M)$ with $M = v/a(z)$. Because of the presence of v, this expression is a function of $\dot{x} = \dot{\xi}$, $\dot{y} = \dot{y}_0 + \dot{\eta}$, $\dot{z} = \dot{z}_0 + \dot{\zeta}$ and through the occurrence of the functions $\rho(z)$ and $a(z)$ it is also a function of $z = z_0 + \zeta$. On the assumption that ξ, η, ζ and their derivatives are small, $\rho(z)vC(M)$ at any time will be nearly equal to $\rho(z_0)v_0C(M_0)$ whose arguments pertain to the range table trajectory. If $\rho(z)vC(M)$ is expanded in a power series in $\dot{\xi}$, $\dot{\eta}$, $\dot{\zeta}$, ζ, the first terms of the expansion are

$$\rho(z)vC(M) = \rho(z_0)v_0C(M_0) + M_0\left[\rho'(z_0)a(z_0)C(M_0) - \rho(z_0)a'(z_0)M_0C'(M_0)\right]\zeta$$
$$+ \rho(z_0)\left[C(M_0) + M_0C'(M_0)\right]\frac{\dot{y}_0}{v_0}\dot{\eta} + \rho(z_0)\left[C(M_0)\right. \tag{9}$$
$$\left. + M_0C'(M_0)\right]\frac{\dot{z}_0}{v_0}\dot{\zeta} + \dots ,$$

the neglected terms involving products and powers of $\dot{\xi}$, $\dot{\eta}$, $\dot{\zeta}$, ζ. The primes in this expression denote derivatives, not with respect to time, but with respect to the indicated arguments of the

functions concerned. Substituting this expansion in equations (8) and again neglecting products and powers of ξ, η, ζ, ζ,

$$\ddot{\xi} = -\frac{A}{2m}\rho(z_0)v_0C(M_0)\dot{\xi} + 2w\dot{\eta}\sin\phi - 2w\dot{\zeta}\cos\phi\cos B + 2w\dot{y}_0\sin\phi - 2w\dot{z}_0\cos\phi\cos B.$$

$$\begin{aligned}\ddot{\eta} = &-2w\dot{\xi}\sin\phi - \frac{A}{2m}\left\{\rho(z_0)v_0C(M_0) + \rho(z_0)\left[C(M_0) + M_0C'(M_0)\right]\frac{\dot{y}_0^2}{v_0}\right\}\dot{\eta}\\ &-\left\{\frac{A}{2m}\rho(z_0)\left[C(M_0) + M_0C'(M_0)\right]\frac{\dot{y}_0\dot{z}_0}{v_0} + 2w\cos\phi\sin B\right\}\dot{\zeta}\\ &-\frac{A}{2m}M_0\left[\rho'(z_0)a(z_0)C(M_0) - \rho(z_0)a'(z_0)M_0C'(M_0)\right]\dot{y}_0\zeta\\ &-2w\dot{z}_0\cos\phi\sin B,\end{aligned} \tag{10}$$

$$\begin{aligned}\ddot{\zeta} = &+2w\dot{\xi}\cos\phi\cos B - \left\{\frac{A}{2m}\rho(z_0)\left[C(M_0) + M_0C'(M_0)\right]\frac{\dot{y}_0\dot{z}_0}{v_0}\right.\\ &\left.-2w\cos\phi\sin B\right\}\dot{\eta}\\ &-\frac{A}{2m}\left\{\rho(z_0)v_0C(M_0) + \rho(z_0)\left[C(M_0) + M_0C'(M_0)\right]\frac{\dot{z}_0^2}{v_0}\right\}\dot{\zeta}\\ &-\frac{A}{2m}M_0\left[\rho'(z_0)a(z_0)C(M_0) - \rho(z_0)a'(z_0)M_0C'(M_0)\right]\dot{z}_0\zeta\\ &+2w\dot{y}_0\cos\phi\sin B.\end{aligned}$$

The quantities with subscript zero refer to the trajectory which does not include the effect of the earth's rotation; they can be obtained from the range tables and hence can be regarded as known functions of the time. Equations (10) are then linear differential equations in ξ, η, ζ with coefficients functions of the time. Owing to the presence of the terms which do not involve ξ, η, ζ, ζ, the equations are nonhomogeneous. If these terms are omitted and the general solution of the resulting equations determined, the method of variation of parameters allows the nonhomogeneous terms to be taken into account. With the equations in their present form, this is not feasible because the occurrence of functions of ϕ and B in the coefficients of ξ, η, ζ would necessitate a separate general solution for each combination of these quantities, and the linearity of the equations would be of little advantage as compared with a direct integration of equation (6).

18. This difficulty can be overcome by making a further approximation in equations (10). This approximation consists of neglecting functions of ϕ and B where they occur as coefficients of $\dot{\xi}$, $\dot{\eta}$, $\dot{\zeta}$, and the justification for this is the fact that a term such as $2w\dot{\zeta}\cos\phi\cos B$ is small relative to $2w\dot{z}_o\cos\phi\cos B$ which is retained. The resulting equations are

$$
\begin{aligned}
\ddot{\xi} &= P_1\dot{\xi} + X \\
\ddot{\eta} &= Q_1\dot{\eta} + Q_2\dot{\zeta} + Q_3\zeta + Y \\
\ddot{\zeta} &= R_1\dot{\eta} + R_2\dot{\zeta} + R_3\zeta + Z
\end{aligned}
\tag{11}
$$

where

$$
\begin{aligned}
P_1 &= -\frac{A}{2m}\rho(z_o)v_oC(M_o) \\
Q_1 &= -\frac{A}{2m}\left\{\rho(z_o)v_oC(M_o) + \rho(z_o)\left[C(M_o) + M_oC'(M_o)\right]\frac{\dot{y}_o^2}{v_o}\right\} \\
Q_2 &= -\frac{A}{2m}\rho(z_o)\left[C(M_o) + M_oC'(M_o)\right]\frac{\dot{y}_o\dot{z}_o}{v_o} = R_1 \\
Q_3 &= -\frac{A}{2m}M_o\left[\rho'(z_o)a(z_o)C(M_o) - \rho(z_o)a'(z_o)M_oC'(M_o)\right]\dot{y}_o \\
R_1 &= -\frac{A}{2m}\rho(z_o)\left[C(M_o) + M_oC'(M_o)\right]\frac{\dot{y}_o\dot{z}_o}{v_o} = Q_2 \\
R_2 &= -\frac{A}{2m}\left\{\rho(z_o)v_oC(M_o) + \rho(z_o)\left[C(M_o) + M_oC'(M_o)\right]\frac{\dot{z}_o^2}{v_o}\right\} \\
R_3 &= -\frac{A}{2m}M_o\left[\rho'(z_o)a(z_o)C(M_o) - \rho(z_o)a'(z_o)M_oC'(M_o)\right]\dot{z}_o \\
X &= +2w\dot{y}_o\sin\phi - 2w\dot{z}_o\cos\phi\cos B \\
Y &= -2w\dot{z}_o\cos\phi\sin B \\
Z &= +2w\dot{y}_o\cos\phi\sin B
\end{aligned}
\tag{12}
$$

FIRST APPROXIMATE SOLUTION OF THE LINEAR EQUATIONS

19. The coefficients $P_1, \ldots R_3$ in equations (11) are functions of quantities which pertain to the range table trajectory and could be computed as functions of the time without regard to the values of ϕ and B. The general solution of the homogeneous equations corresponding to (11) could then be obtained by numerical methods and the solution of the nonhomogeneous equations obtained by variation of parameters. A method based on this procedure will be described presently. First, however, it seems desirable to consider an approximation to the solution of equations (11) which might be sufficiently accurate for some purposes. This solution is obtained simply by neglecting all terms on the right of (11) except X, Y, Z. This is equivalent to assuming that the air resistance which acts on the projectile which is deflected by the earth's rotation is at all times exactly equal to the air resistance acting on the range table projectile. Using the expressions in (12) for X, Y, Z, the solution of the equations is obtained by two integrations of each equation and is

$$
\begin{aligned}
\xi &= \lambda_1(t) \cos\phi \cos B + \lambda_2(t) \sin\phi \\
\eta &= \mu(t) \cos\phi \sin B \\
\zeta &= \nu(t) \cos\phi \sin B
\end{aligned}
\tag{13}
$$

where

$$
\begin{aligned}
\lambda_1(t) &= \mu(t) = -2w \int_0^t z_0 \, dt \\
\lambda_2(t) &= \nu(t) = +2w \int_0^t y_0 \, dt \, .
\end{aligned}
\tag{14}
$$

For each range table trajectory, the functions $\mu(t)$ and $\nu(t)$ can be computed easily from quantities contained in the tables and arranged as auxiliary columns in the range tables. From these auxiliary quantities the components of the deflection due to the earth's rotation can be found from equations (13) for any latitude ϕ and gun bearing B. Equations (13) also appear to indicate a simple method of incorporating a correction for the earth's rotation into the computer. It will be seen presently that the exact solution of equations (11) has the form of equations (13) with coefficient functions of the time which are more complicated than are equations (14), so that while the exact solution is more laborious to compute it offers little more difficulty in application than does the approximate solution.

SECOND SOLUTION OF THE LINEAR EQUATIONS

20. Before attempting to solve equations (11), it is desirable for reference purposes to write the scalar equations equivalent to (1). These are

$$\ddot{x}_o = 0$$

$$\ddot{y}_o = -\frac{A}{2m}\,\rho(z_o)v_oC(M_o)\dot{y}_o \qquad = P_1\dot{y}_o \qquad (15)$$

$$\ddot{z}_o = -\frac{A}{2m}\,\rho(z_o)v_oC(M_o)\dot{z}_o - g = P_1\dot{z}_o - g\,,$$

where P_1 has the same definition as in (12).

21. Considering now equations (11), the first equation is evidently independent of the other two and can be treated separately. To apply the method of variation of parameters we first write this as a normal system of first order equations,

$$\frac{d\xi}{dt} = \dot{\xi}\,,$$

$$\frac{d\dot{\xi}}{dt} = P_1\dot{\xi} + X\,, \qquad (16)$$

and then attempt to find the general solution of the corresponding homogeneous equations

$$\frac{d\xi}{dt} = \dot{\xi}\,,$$

$$\frac{d\dot{\xi}}{dt} = P_1\dot{\xi}\,. \qquad (17)$$

We notice first that if b_1 is an arbitrary constant, then $\xi = b_1$, $\dot{\xi} = 0$ is a particular solution of (17). Next, the second of equations (15) could be written as an equivalent system of normal equations which would be identical with (17). Since the equations obtained from (15) are satisfied by the range table value of y_o, the functions $\xi = y_o$, $\dot{\xi} = \dot{y}_o$ constitute a second particular solution of (17). As a consequence of the linearity of equations (17), their general

solution is therefore

$$
\begin{aligned}
\xi &= b_1 + b_2 y_o \\
\dot{\xi} &= b_2 \dot{y}_o \, .
\end{aligned}
\tag{18}
$$

To obtain the solution of the nonhomogeneous equations (16), b_1 and b_2 are regarded as functions of the time and equations (18) as a change of dependent variables from ξ, $\dot{\xi}$ to b_1, b_2 to be made in equations (16). Making this substitution and using the fact that equations (18) satisfy (17),

$$\frac{db_1}{dt} + \frac{db_2}{dt} y_o = 0$$

$$\frac{db_2}{dt} \dot{y}_o = X.$$

Recalling that X is a known function of the time, the general solution of these equations is

$$b_1 = -\int_o^t \frac{y_o X}{\dot{y}_o} dt + b_{10}$$

$$b_2 = +\int_o^t \frac{X}{\dot{y}_o} dt + b_{20} \, ,$$

where b_{10} and b_{20} are arbitrary constants. Substituting these values of b_1 and b_2 in (18) and determining b_{10} and b_{20} so that $\xi = \dot{\xi} = 0$ at $t = 0$, the solution of (16) is then

$$\xi = y_o \int_o^t \frac{X}{\dot{y}_o} dt - \int_o^t \frac{y_o X}{\dot{y}_o} dt$$

$$\dot{\xi} = \dot{y}_o \int_o^t \frac{X}{\dot{y}_o} dt \, .$$

Substituting in the first of these equations the expression for X in (12),

$$\xi = 2w\left[\int_0^t \frac{y_0 \dot{z}_0}{\dot{y}_0}\,dt - y_0 \int_0^t \frac{\dot{z}_0}{\dot{y}_0}\,dt\right] \cos \phi \cos B + 2w\left[y_0 t - \int_0^t y_0\,dt\right] \sin \phi , \tag{19}$$

and this has the same form as the first of (13) with, however, coefficients which are more complicated functions of the time.

22. To obtain η and ζ, let the last two of equations (11) be written as the normal system

$$\begin{aligned}
\frac{d\eta}{dt} &= \dot{\eta} \\
\frac{d\dot{\eta}}{dt} &= Q_1\dot{\eta} + Q_2\dot{\zeta} + Q_3\zeta + Y \\
\frac{d\zeta}{dt} &= \dot{\zeta} \\
\frac{d\dot{\zeta}}{dt} &= R_1\dot{\eta} + R_2\dot{\zeta} + R_3\zeta + Z.
\end{aligned} \tag{20}$$

To apply the method of variation of parameters, we wish first to find the general solution of the corresponding homogeneous equations

$$\begin{aligned}
\frac{d\eta}{dt} &= \dot{\eta} \\
\frac{d\dot{\eta}}{dt} &= Q_1\dot{\eta} + Q_2\dot{\zeta} + Q_3\zeta \\
\frac{d\zeta}{dt} &= \dot{\zeta} \\
\frac{d\dot{\zeta}}{dt} &= R_1\dot{\eta} + R_2\dot{\zeta} + R_3\zeta ,
\end{aligned} \tag{21}$$

and this latter solution can be written as a linear combination of four independent particular solutions. Such particular solutions can be obtained from an interpretation of equations (21) arising from a problem which is quite different from that being considered here. Consider equation (1) together with the "normal" initial values of $\underline{r_0}$ and $\underline{r_0'}$ which define the particular range table trajectory under consideration, the value of r_0 being of course zero while $\underline{r_0'}$ is the initial

velocity. Consider also equation (6) with w set equal to zero together with "abnormal" initial values of $\underline{r}$ and $\underline{r}'$ which are slightly different from $\underline{r}_o$ and $\underline{r}_o'$, respectively. The solution of equation (6) therefore gives the trajectory of a projectile, unaffected by the earth's rotation, which is launched with abnormal initial conditions differing slightly from those defining the range table trajectory. Again defining $\Delta = \underline{r} - \underline{r}_o$, equation (7) with $w = 0$ is the differential equation defining the position of the abnormal projectile relative to the normal projectile; equations (8) with $w = 0$ are the scalar equivalents of (7); equations (10) with $w = 0$, are the linear approximations to equations (8); and, with $w = 0$, i.e., with $X = Y = Z = 0$, equations (11) are simply an abbreviated form of (8); so that equations (11) are the linearized equations defining the distance components between the normal and abnormal projectiles. But equations (21) are equivalent to the second and third of equations (11) with $w = 0$. It is concluded, therefore, that equations (21) with suitable initial conditions define the distance components between a normal (range table) projectile and an abnormal projectile in the same plane which differs from the normal projectile only in being launched with different initial conditions, the earth's rotation not being considered in either case.

23. As a special abnormal trajectory, consider one for which the initial velocity is the same as that of the normal trajectory but the point of launching is displaced horizontally a distance $\eta = 1$ in the common plane of fire. Throughout its flight the abnormal projectile will simply be displaced horizontally a distance $\eta = 1$ from the normal projectile, and hence the abnormal trajectory is defined by the quantities

$$\eta = 1, \; \dot{\eta} = 0, \; \zeta = 0, \; \dot{\zeta} = 0 . \qquad (22)$$

That these quantities do, in fact, constitute a particular solution of equations (21) is seen by substitution.

24. To obtain a second particular solution, we consider an abnormal projectile which is launched from the same point and with the same velocity as the normal projectile, but at a small time δt earlier. It is clear that both projectiles will follow the same path in the plane of fire but that the abnormal projectile will be displaced from the normal projectile by component distances equal to the components of travel of the normal projectile in time δt. The distance and velocity components of the abnormal projectile relative to the normal projectile, for δt sufficiently small, are then

$$\eta = \dot{y}_o \delta t, \; \dot{\eta} = \ddot{y}_o \delta t, \; \zeta = \dot{z}_o \delta t, \; \dot{\zeta} = \ddot{z}_o \delta t,$$

and these functions must constitute a solution of equations (21). Since the differential equations are linear, the proportional quantities

$$\eta = \dot{y}_0, \ \dot{\eta} = \ddot{y}_0, \ \zeta = \dot{z}_0, \ \dot{\zeta} = \ddot{z}_0$$

must also be a solution. Although the second derivatives $\ddot{y}_0$ and $\ddot{z}_0$ are not given in the range tables, they can be expressed in terms of quantities which are in the range tables by means of equations (15). Making these substitutions,

$$\eta = \dot{y}_0, \ \dot{\eta} = P_1\dot{y}_0, \ \zeta = \dot{z}_0, \ \dot{\zeta} = P_1\dot{z}_0 - g \ . \qquad (23)$$

That these functions actually do constitute a particular solution of equations (21) can be verified by substitution.

25. To obtain the general solution of (21) we require two more particular solutions. There are no other particular solutions which are as easily obtained as (22) and (23) and it would ordinarily be necessary to obtain the necessary additional solutions by numerical integration of equations (21). However, the AAF range tables are sufficiently complete that quantities can be obtained from them which will approximately satisfy the equations. To obtain such quantities we continue with the interpretation of (21) as equations defining the variations from a normal trajectory due to abnormal initial conditions, the rotation of the earth not being considered. Suppose that the normal trajectory is defined by values of gun elevation and muzzle velocity for which a range table trajectory is tabulated. Suppose further that the abnormal trajectory is defined by the same muzzle velocity (actually a scalar: speed) but a gun elevation one degree greater. These abnormalities in the initial velocity (vector) could be expressed in terms of initial values of $\dot{\eta}$ and $\dot{\zeta}$ and the resulting solution of equations (21) would give approximately the increments of distance and velocity components in passing from the normal trajectory to the abnormal trajectory. Since, however, the range tables contain trajectories for every degree of gun elevation at the normal muzzle velocity of 4000 ft/sec, the exact values of these increments can be found by taking the differences of tabulated quantities pertaining to two range table trajectories, and these differences should constitute approximate solutions of equations (21). Using subscript 0, as before, to refer to the normal trajectory and subscript 3 (which is intended to suggest the third particular solution) to refer to the range table trajectory having a gun elevation one degree greater, the approximate solution obtained by this process can be written

$$\eta = y_3 - y_0 = \eta_3, \ \dot{\eta} = \dot{y}_3 - \dot{y}_0 = \dot{\eta}_3, \ \zeta = z_3 - z_0 = \zeta_3,$$
$$\dot{\zeta} = \dot{z}_3 - \dot{z}_0 = \dot{\zeta}_3 \ . \qquad (24)$$

Some indication of the accuracy of this solution can be obtained by noticing that the increments of range table quantities for a two degree increase in gun elevation are very nearly double the corresponding increments for a one degree increase.

[illegible]. A fourth particular solution of equations (21) can be obtained approximately by considering the abnormal trajectory having the same gun elevation as the normal trajectory but a muzzle velocity 100 ft/sec greater. This solution is

$$\eta = y_4 - y_{\text{[illegible]}} = \eta_4,\quad \dot{\eta} = \dot{y}_4 - \dot{y}_{\text{[illegible]}} = \dot{\eta}_4,\quad \zeta = z_4 - z_{\text{[illegible]}} = \zeta_4,\quad \dot{\zeta} = \dot{z}_4 - \dot{z}_{\text{[illegible]}} = \dot{\zeta}_4, \tag{25}$$

and the subscript 4 refers to the range table trajectory having a muzzle velocity of 4100 ft/sec. Here again, an examination of the tabulated quantities for muzzle velocity increments of [illegible] ft/sec and 200 ft/sec suggests that this solution is not [illegible] in error.

[illegible]. While the first three particular solutions can be obtained for normal trajectories at every degree of gun elevation with a muzzle velocity of 4000 ft/sec, the fact that the range tables contain 4100 ft/sec trajectories only at every five degrees of gun elevation allows the fourth particular solution (24) to be obtained only at these values of gun elevation. It seems likely that the correction for the earth's [illegible] changes sufficiently smoothly with gun elevation that auxiliary quantities computed at every five degrees of gun elevation would be adequate.

[illegible]. Having determined four particular solutions (22), [illegible], (25) of equations (21), the general solution is

$$\begin{aligned}
\eta &= c_1 \cdot 1 + \text{[illegible]} + c_3\eta_3 + c_4\eta_4 \\
\dot{\eta} &= \text{[illegible]} + c_3\dot{\eta}_3 + c_4\dot{\eta}_4 \\
\zeta &= c_1 \cdot 0 + c_2\,\text{[illegible]} + c_3\zeta_3 + c_4\zeta_4 \\
\dot{\zeta} &= c_1\,\text{[illegible]} + c_2(\text{[illegible]}) + c_3\dot{\zeta}_3 + c_4\dot{\zeta}_4
\end{aligned} \tag{[illegible]}$$

[illegible], c_2, [illegible] are arbitrary constants. We now regard [illegible], c_4 as functions of the time and think of equations [illegible] as a change of dependent variables from η, $\dot{\eta}$, ζ, $\dot{\zeta}$ to c_1, c_2, c_3, c_4 which is to be made in the nonhomogeneous equations (20), the fact that equations (2[illegible]) with constant

coefficients satisfy equations (21) being of incidental benefit in simplifying the resulting equations. The result of this substitution and simplification is

$$
\begin{aligned}
\frac{dc_1}{dt} + \dot{y}_o\frac{dc_2}{dt} + \eta_3\frac{dc_3}{dt} + \eta_4\frac{dc_4}{dt} &= 0 \\
F_1\dot{y}_o\frac{dc_2}{dt} + \dot{\eta}_3\frac{dc_3}{dt} + \dot{\eta}_4\frac{dc_4}{dt} &= Y \\
\dot{z}_o\frac{dc_2}{dt} + \zeta_3\frac{dc_3}{dt} + \zeta_4\frac{dc_4}{dt} &= 0 \\
(F_1\dot{z}_o - g)\frac{dc_2}{dt} + \dot{\zeta}_3\frac{dc_3}{dt} + \dot{\zeta}_4\frac{dc_4}{dt} &= Z\ .
\end{aligned}
\tag{27}
$$

To solve these equations for the derivatives $\frac{dc_1}{dt}$, etc. it is convenient to have the determinant of their coefficients

$$
D(t) = \begin{vmatrix} 1 & \dot{y}_o & \eta_3 & \eta_4 \\ 0 & F_1\dot{y}_o & \dot{\eta}_3 & \dot{\eta}_4 \\ 0 & \dot{z}_o & \zeta_3 & \zeta_4 \\ 0 & F_1\dot{z}_o - g & \dot{\zeta}_3 & \dot{\zeta}_4 \end{vmatrix} .
$$

The value of $D(t)$ as a function of t could be computed directly from its definition using quantities tabulated in the range tables, but it is simpler to use a property of such determinants which is proved in F.R. Moulton's *Differential Equations*, p. 23[illegible]. [illegible] = 0, $D(t)$ reduces to

$$D(0) = \begin{vmatrix} 1 & \dot{y}_o & 0 & 0 \\ 0 & P_1\dot{y}_o & \dot{\eta}_3 & \dot{\eta}_4 \\ 0 & \dot{z}_o & 0 & 0 \\ 0 & P_1\dot{z}_o - g & \dot{\zeta}_3 & \dot{\zeta}_4 \end{vmatrix} = -\dot{z}_o \begin{vmatrix} \dot{y}_3 - \dot{y}_o & \dot{y}_4 - \dot{y}_o \\ \dot{z}_3 - \dot{z}_o & \dot{z}_4 - \dot{z}_o \end{vmatrix},$$

where the quantities P_1, $\dot{y}_o$, etc. have their initial values. If E is the gun elevation angle for the normal trajectory under consideration and v_o is taken for this purpose as the muzzle velocity, then these initial values are

$$\begin{aligned} \dot{y}_o &= v_o \cos E \\ \dot{z}_o &= v_o \sin E \\ \dot{y}_3 &= v_o \cos (E + 1^\circ) \\ \dot{z}_3 &= v_o \sin (E + 1^\circ) \\ \dot{y}_4 &= (v_o + 100) \cos E \\ \dot{z}_4 &= (v_o + 100) \sin E \end{aligned}$$

and $D(0)$ becomes

$$D(0) = -v_o \sin E \begin{vmatrix} v_o\left[\cos (E + 1^\circ) - \cos E\right] & 100 \cos E \\ v_o\left[\sin (E + 1^\circ) - \sin E\right] & 100 \sin E \end{vmatrix} = 100 v_o^2 \sin 1^\circ \sin E$$

or with $v_o = 4000$ ft/sec,

$$D(0) = 2.792385 \times 10^7 \sin E . \qquad (28)$$

29. It is convenient at this point to consider also the initial values of c_1, c_2, c_3, c_4 which must be imposed on the

solution of (27). For the deflection produced by the earth's rotation, $\eta = \dot{\eta} = \zeta = \dot{\zeta} = 0$ at $t = 0$. Putting $t = 0$ in (26) we therefore obtain four simultaneous linear equations for the initial values of c_1, c_2, c_3, c_4. The determinant of the coefficients of these equations is $D(0)$ and, since $D(0) \neq 0$ and the equations are homogeneous, the initial values of c_1, c_2, c_3, c_4 are all zero.

30. Returning to the evaluation of $D(t)$, the property proved by Moulton, when applied to this problem is

$$D(t) = D(0) \exp \int_0^t (Q_1 + R_2)\, dt. \tag{29}$$

From the definitions of Q_1 and R_2 in (12),

$$Q_1 + R_2 = -\frac{A}{2m}\left[3\rho(z_0)v_0 C(M_0) + \rho(z_0)v_0 M_0 C'(M_0)\right], \tag{30}$$

which can be evaluated from quantities contained in the range tables and $D = D(t)$ computed as a function of t. The solution of equations (27) for $\frac{dc_1}{dt}$, etc. is then

$$\frac{dc_1}{dt} = \frac{1}{D}\begin{vmatrix} 0 & \dot{y}_0 & \eta_3 & \eta_4 \\ Y & P_1\dot{y}_0 & \dot{\eta}_3 & \dot{\eta}_4 \\ 0 & \dot{z}_0 & \zeta_3 & \zeta_4 \\ Z & P_1\dot{z}_0 - g & \dot{\zeta}_3 & \dot{\zeta}_4 \end{vmatrix}$$

$$\frac{dc_2}{dt} = \frac{1}{D}\begin{vmatrix} 1 & 0 & \eta_3 & \eta_4 \\ 0 & Y & \dot{\eta}_3 & \dot{\eta}_4 \\ 0 & 0 & \zeta_3 & \zeta_4 \\ 0 & Z & \dot{\zeta}_3 & \dot{\zeta}_4 \end{vmatrix}$$

$$\frac{dc_3}{dt} = \frac{1}{D} \begin{vmatrix} 1 & \dot{y}_0 & 0 & \eta_4 \\ 0 & p_1\dot{y}_0 & Y & \dot{\eta}_4 \\ 0 & \dot{z}_0 & 0 & \zeta_4 \\ 0 & p_1\dot{z}_0 - g & Z & \dot{\zeta}_4 \end{vmatrix}$$

$$\frac{dc_4}{dt} = \frac{1}{D} \begin{vmatrix} 1 & \dot{y}_0 & \eta_3 & 0 \\ 0 & p_1\dot{y}_0 & \dot{\eta}_3 & Y \\ 0 & \dot{z}_0 & \zeta_3 & 0 \\ 0 & p_1\dot{z}_0 - g & \dot{\zeta}_3 & Z \end{vmatrix}$$

Substituting the expressions for Y and Z given in (12), these become

$$\frac{dc_1}{dt} = \frac{2w}{D} \begin{vmatrix} 0 & \dot{y}_0 & \eta_3 & \eta_4 \\ -\dot{z}_0 & p_1\dot{y}_0 & \dot{\eta}_3 & \dot{\eta}_4 \\ 0 & \dot{z}_0 & \zeta_3 & \zeta_4 \\ +\dot{y}_0 & p_1\dot{z}_0 - g & \dot{\zeta}_3 & \dot{\zeta}_4 \end{vmatrix} \cos\phi \sin B$$

$$\frac{dc_2}{dt} = \frac{2w}{D} \begin{vmatrix} 1 & 0 & \eta_3 & \eta_4 \\ 0 & -\dot{z}_0 & \dot{\eta}_3 & \dot{\eta}_4 \\ 0 & 0 & \zeta_3 & \zeta_4 \\ 0 & +\dot{y}_0 & \dot{\zeta}_3 & \dot{\zeta}_4 \end{vmatrix} \cos\phi \sin B$$

$$\frac{dc_3}{dt} = \frac{2w}{D} \begin{vmatrix} 1 & \dot{y}_0 & 0 & \eta_4 \\ 0 & p_1\dot{y}_0 & -\dot{z}_0 & \dot{\eta}_4 \\ 0 & \dot{z}_0 & 0 & \zeta_4 \\ 0 & p_1\dot{z}_0 - g & +\dot{y}_0 & \dot{\zeta}_4 \end{vmatrix} \cos\phi \sin B$$

$$\frac{dc_4}{dt} = \frac{2w}{D}\begin{vmatrix} 1 & y_o & \eta_3 & 0 \\ 0 & P_1\dot{y}_o & \dot{\eta}_3 & -\dot{z}_o \\ 0 & \dot{z}_o & \zeta_3 & 0 \\ 0 & P_1\dot{z}_o - g & \dot{\zeta}_3 & +\dot{y}_o \end{vmatrix} \cos \phi \sin B \quad .$$

By expanding the determinants, the integrals of these expressions can be written

$$\begin{aligned} c_1 &= 2w\, \psi_1(t) \cos \phi \sin B \\ c_2 &= 2w\, \psi_2(t) \cos \phi \sin B \\ c_3 &= 2w\, \psi_3(t) \cos \phi \sin B \\ c_4 &= 2w\, \psi_4(t) \cos \phi \sin B \; , \end{aligned} \qquad (31)$$

where

$$\psi_1(t) = \int_o^t \frac{1}{D}\Bigg[(P_1 v_o^2 - g\dot{z}_o)\begin{vmatrix} \eta_3 & \eta_4 \\ \zeta_3 & \zeta_4 \end{vmatrix} - \dot{y}_o^2 \begin{vmatrix} \dot{\eta}_3 & \dot{\eta}_4 \\ \zeta_3 & \zeta_4 \end{vmatrix}$$

$$- \dot{y}_o\dot{z}_o\left(\begin{vmatrix} \eta_3 & \eta_4 \\ \dot{\eta}_3 & \dot{\eta}_4 \end{vmatrix} - \begin{vmatrix} \zeta_3 & \zeta_4 \\ \dot{\zeta}_3 & \dot{\zeta}_4 \end{vmatrix} \right) - \dot{z}_o^2 \begin{vmatrix} \eta_3 & \eta_4 \\ \dot{\zeta}_3 & \dot{\zeta}_4 \end{vmatrix} \Bigg] dt$$

$$\psi_2(t) = \int_o^t \frac{1}{D}\left[\dot{y}_o \begin{vmatrix} \dot{\eta}_3 & \dot{\eta}_4 \\ \zeta_3 & \zeta_4 \end{vmatrix} - \dot{z}_o \begin{vmatrix} \zeta_3 & \zeta_4 \\ \dot{\zeta}_3 & \dot{\zeta}_4 \end{vmatrix} \right] dt \qquad (32)$$

$$\psi_3(t) = \int_o^t \frac{1}{D}\left[\dot{z}_o \begin{vmatrix} \dot{z}_o & \zeta_4 \\ P_1 z_o - g & \dot{\zeta}_4 \end{vmatrix} - \dot{y}_o \begin{vmatrix} P_1\dot{y}_o & \dot{\eta}_4 \\ \dot{z}_o & \zeta_4 \end{vmatrix} \right] dt$$

$$\psi_4(t) = \int_o^t \frac{1}{D}\left[\dot{y}_o \begin{vmatrix} P_1\dot{y}_o & \dot{\eta}_3 \\ \dot{z}_o & \zeta_3 \end{vmatrix} - \dot{z}_o \begin{vmatrix} \dot{z}_o & \zeta_3 \\ P_1\dot{z}_o - g & \dot{\zeta}_3 \end{vmatrix} \right] dt \quad .$$

Substituting c_1, c_2, c_3, c_4 from (31) into (26), the particular solution of equations (20) required for the problem of the earth's rotation is obtained. The resulting expressions for η and ζ, together with the expression (19) for ξ, can again be written in the form (13),

$$\begin{aligned} \xi &= \lambda_1(t) \cos \phi \cos B + \lambda_2(t) \sin \phi \\ \eta &= \mu(t) \cos \phi \sin B \\ \zeta &= \nu(t) \cos \phi \sin B, \end{aligned} \tag{13}$$

where, now,

$$\lambda_1(t) = 2w\left[\int_0^t \frac{y_0 \dot{z}_0}{\dot{y}_0}\, dt - y_0 \int_0^t \frac{\dot{z}_0}{\dot{y}_0}\, dt\right]$$

$$\lambda_2(t) = 2w\left[y_0 t - \int_0^t y_0\, dt\right] \tag{33}$$

$$\mu(t) = 2w\left[\psi_1(t) + \dot{y}_0\, \psi_2(t) + \eta_3\, \psi_3(t) + \eta_4\, \psi_4(t)\right]$$

$$\nu(t) = 2w\left[\qquad \dot{z}_0\, \psi_2(t) + \zeta_3\, \psi_3(t) + \zeta_4\, \psi_4(t)\right].$$

In addition to the coordinate increments produced by the earth's rotation, the Guidance Computer should, in principle, take account of the velocity increments $\dot{\xi}$, $\dot{\eta}$, $\dot{\zeta}$. These could easily be expressed in terms of quantities already derived, but it seems likely that it is unnecessary in practice to take into account the influence of the earth's rotation on the velocity components of the AAM.

31. To evaluate numerically the integrals in (32) and (33), we require the position and velocity components for the normal trajectory, the differences between these and the

corresponding quantities for the two abnormal trajectories, and the values of P_1 and D for the normal trajectory, D being obtained in terms of the integral of $Q_1 + R_2$.

APPLICATION TO AAP TRAJECTORIES

32. Up to this point no assumptions have been made regarding the functions $\rho(z)$, $a(z)$, and $C(M)$. The particular form of $C(M)$ on which the preliminary range tables are based is $C(M) = k_1 M^{-k_2}$, from which $M\,C'(M) = -k_2 C(M)$. Substituting this in the second term of (30),

$$Q_1 + R_2 = -\frac{A}{2m}(3-k_2)\,\rho(z_o)v_o C(M_o) = (3-k_2)P_1 .$$

To evaluate P_1 we consider also the functions $\rho(z)$ and $a(z)$. The preliminary range tables are based on the NACA Standard Atmosphere for which $\rho(z) = \rho(0)(1 - k_3 z)^{k_4}$ and $a(z) = a(0)\cdot(1 - k_3 z)^{1/2}$. Hence

$$P_1 = -\frac{A}{2m}\rho(z_o)v_o C(M_o) = -\frac{A\rho(o)}{2m}\cdot k_1[a(o)]^{k_2}(1 - k_3 z)^{k_4 + \frac{k_2}{2}} v_o^{1-k_2} .$$

33. For the present design of Angled Arrow Projectile.

$$A = \frac{\pi}{4}\left(\frac{4.45}{12}\right)^2 \qquad (\text{ft}^2)$$

$$m = 2.261145 \qquad (\text{slug})$$

(Note: The value of m has been adjusted to remove an inconsistency in NAVORD Report 2160. A smaller inconsistency still remains in the derived quantities which follow. The preliminary range tables were computed with the expression for P_1 which is given below, and the constants should be further adjusted to reduce the numerical coefficient in this expression.)

$$k_1 = 0.66061$$

$$k_2 = 0.6775 ,$$

and for the NACA Standard Atmosphere,

$$k_3 = 6.87919 \times 10^{-6} \quad (ft^{-1})$$

$$k_4 = 4.255$$

$$\rho(0) = 0.002378 \quad (slug/ft^3)$$

$$a(0) = 1117 \quad (ft/sec) \quad .$$

With these values,

$$\frac{A\rho(o)}{2m} = 5.67938 \times 10^{-5}$$

$$\frac{A\rho(o)}{2m} k_1 [a(o)]^{k_2} = 4.359495 \times 10^{-3}$$

and the expression for P_1 becomes

$$P_1 = -4.359495 \times 10^{-3}(1 - 6.87919 \times 10^{-6} z_c)^{4.59375} v_o^{0.3225} \quad .$$

EXAMPLE

34. As a numerical example of the computation of the effect of the earth's rotation, an AAP trajectory with a gun elevation of 30° and a muzzle velocity of 4000 ft/sec has been considered. The auxiliary functions needed for the application of equations (13) have been computed by the approximate formulas (14) and by the more exact formulas (33) and are tabulated in Table 1. Some explanation is necessary for the evident roughness of the values computed from formulas (33). Part of this roughness arises from the necessity of forming the differences η_2 etc. between range table trajectories. While the coordinates and velocities in the preliminary range table are tabulated with sufficient accuracy for the principal uses of the table, this accuracy is not great enough to produce smooth values of the necessary differences. Since the tabulated values have been rounded off, it would be preferable to obtain the differences η_2 etc. from the unrounded values. It is understood

that the numerical integrations for the preliminary range table were conducted in such a manner that the unrounded values have not been preserved, so that this refinement could not be made in the present computation. If, when the final range table is prepared, it has been decided to correct for the earth's rotation by means of equations (13) and (33), then the numerical integration could probably be arranged so that the required differences would be available and formulas (33) might be evaluated at the same time. Another source of roughness in these quantities was the malfunctioning of the calculating machine which was used in evaluating formulas (33) and was not detected until the work was nearly finished. In the presence of the unavoidable roughness due to insufficient accuracy of the differences, it did not seem worth while to repeat the entire calculation.

35. To exhibit the effects of the earth's rotation on the coordinates of the projectile, both sets of functions in Table 1 have been used to compute ξ, η, ζ by means of equations (13) for a latitude $\phi = 45°$ and a gun bearing angle $B = 45°$, the results being given in Table 2. An examination of the table shows that there are appreciable differences between the values obtained by the two methods and the question arises whether the values obtained from (13) and (33) can be regarded as correct, or whether the approximations involved in these equations introduce important errors. To answer this question, the component equations corresponding to (5) have been integrated numerically for the initial conditions being considered and from the resulting values x, y, z of the coordinates, the coordinates $x_o = 0$, y_o, z_o of the projectile on the range table trajectory have been subtracted, to produce values of ξ, η, ζ which involve no approximation except for the justifiable neglect of w^2 terms in (5). These values are also given in Table 2, from which it is seen that the approximate formulas (13) and (33) are practically without error.

36. To simplify the comparison between the different methods of computing the effect of the earth's rotation, Table 3 which has been derived from Table 2 gives the magnitude of the remaining error (vector sum of three components) if the earth's rotation correction is made by four methods: not at all, by the use of equations (13) and (14), the use of equations (13) and (33), and by the direct numerical integration of (5), this last method being regarded as exact.

COMPARISON OF METHODS

37. With some apology for attempting to assess such questions in a purely quantitative manner, Table 4 is an attempt to compare the relative costs of the four methods and the values of the corrections obtained from them. In the first row, the last entry is intended to indicate that the method requiring the numerical integration of equation (6) is completely unfeasible for the large number of cases that would have to be treated. The numbers 1 and 20 for the second and third entries are a good indication of the relative times that were required for the computation of the corrections by these two methods. Considering the second row, the comparison between the second and third entries is probably fairly reliable, equations (13) indicating how a correction obtained by either of these methods might be incorporated in the Guidance Computer. The use of equations (14) would require the empirical representation of two functions of time of flight for each range table trajectory, while the representation of the corrections obtained from (33) would require four such functions. The fourth entry in this row is based on the belief that if the corrections were obtained by this method the corresponding modification of the computer would have to be arrived at entirely empirically and might require a complete alteration of the existing ballistic solution. The last row of Table 4 gives an estimate of the relative values of the corrections obtained by the different methods, but ignores the fact that the effect of the earth's rotation, if not corrected, is only one of many other errors that will be present in the complete AAP System, and the removal of this single error might not produce a significant increase in the effectiveness of the system. In comparing this error with other ballistic errors, it should be remembered that the quantities in Table 3 are miss distances of burst positions of time fuzed ammunition. For VT fuzed ammunition, a more appropriate measure of the error is the minimum distance between projectile and target. Since the miss distance produced by neglect of the earth's rotation is approximately normal to the trajectory (the Coriolis acceleration $2\omega \times r'$ being perpendicular to the velocity vector r'), the miss distances for both types of ammunition are nearly equal for target aircraft which are approaching the gun. For certain other ballistic errors (e.g., those due to incorrect muzzle velocity of air density corrections) the VT fuze miss distance for approaching targets is significantly smaller than the time fuze miss distance, and in this respect the uncorrected effect of the earth's rotation is especially detrimental.

RELATED TOPICS

38. When a projectile design has been achieved which is regarded as satisfactory from the points of view of performance and suitability for production, it is intended to manufacture a number of projectiles for use in experimental firings on the results of which the final AAP range tables will be based. While the details of the experimental firings have not been decided upon, the principal observations to be made will furnish for each round a record of projectile position as a function of time. Various secondary observations might also be desirable: time intervals measured by the In-Bore Chronograph, muzzle velocity as determined by some other form of chronograph, photographs to indicate whether the sabot separation is normal, etc.

39. The principal object of these firings will be the verification or revision of the best previously available drag coefficient, whether that drag coefficient is the one used in the computation of the existing preliminary range tables or some improved drag coefficient which might be arrived at later on the basis of theoretical considerations, model experiments, or earlier firings of full scale projectiles. Certain other information should also be obtained from these firings as, for example, whether the angle of departure is equal to the gun elevation angle (they are not equal for the 5"/38 gun).

40. The determination of such unknown factors from the results of experimental firings requires that all known departures from the standard trajectories shall be taken into account. Among these effects are that of the earth's rotation, non-standard atmospheric conditions, etc. By employing methods similar to those used in this Report, it seems likely that a systematic procedure can be developed for allowing for known influences on the standard trajectories and determining the changes in drag coefficient and angle of departure necessary to remove the remaining discrepancies between observations and theory. Such a procedure would probably require the computation, for each range table trajectory, of auxiliary functions somewhat similar to those of equations (33).

41. It seems desirable that these questions should be thoroughly investigated well in advance of the planning of the experimental firings. Such a study would indicate the most useful data which might be obtained from the firings and would allow an early start on the extensive computations which might be necessary to make the most effective use of the observations.

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O. T. SCHULTZ

OTS/es

TABLE 1

AUXILIARY FUNCTIONS

t	Equations (14)		Equations (33)			
	$\lambda_1 = \mu$	$\lambda_2 = \nu$	λ_1	λ_2	μ	ν
sec	ft	ft	ft	ft	ft	ft
0	0.0	0.0	0.0	0.0	0.0	0.0
1	- 0.1	+ 0.2	- 0.1	+ 0.2	- 0.1	+ 0.2
2	0.5	1.0	0.5	0.9	0.4	0.9
3	1.2	2.1	1.2	2.0	1.1	2.1
4	2.1	3.7	2.0	3.5	1.9	3.5
5	- 3.2	+ 5.8	- 3.0	+ 5.3	- 2.9	+ 5.3
6	4.5	8.1	4.1	7.4	4.0	7.4
7	6.1	10.9	5.4	9.7	5.2	9.9
8	7.7	14.0	6.8	12.4	6.5	12.5
9	9.6	17.5	8.3	15.2	7.9	15.5
10	-11.6	+21.3	-10.0	+18.3	- 9.3	+18.7
11	13.8	25.4	11.7	21.6	10.9	22.2
12	16.1	29.9	13.5	25.2	12.3	25.9
13	18.5	34.6	15.3	28.9	13.9	29.7
14	21.0	39.6	17.3	32.8	15.6	33.2
15	-23.7	+44.9	-19.3	+36.8	-17.0	+38.1
16	26.4	50.5	21.3	41.0	18.7	42.3
17	29.3	56.4	23.4	45.4	20.2	47.1
18	32.2	62.5	25.5	50.0	21.8	51.9
19	35.3	68.9	27.7	54.6	23.3	56.9
20	-38.4	+75.7	-29.9	+59.4	-24.9	+61.9
21	41.6	82.4	32.1	64.4	26.1	67.3
22	44.9	89.5	34.3	69.4	27.5	72.7
23	48.1	96.8	36.5	74.6	28.8	78.4
24	51.5	104.4	38.7	79.9	30.1	84.[illegible]
25	-54.[illegible]	+112.2	-40.9	+85.3	-31.3	+89.[illegible]

TABLE 2

COMPARISON OF RESULTS

	(14)			(33)			(6)		
t sec	ξ ft	η ft	ζ ft	ξ ft	η ft	ζ ft	ξ ft	η ft	ζ ft
0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0
2	0	0	+ 1	0	0	0	0	0	0
3	+ 1	- 1	1	+ 1	- 1	+ 1	+ 1	- 1	+ 1
4	2	1	2	1	1	2	2	1	2
5	+ 3	- 2	+ 3	+ 2	- 1	+ 3	+ 2	- 2	+ 2
6	3	2	4	3	2	4	3	2	[illegible]
7	5	3	5	4	3	5	4	2	5
8	[illegible]	4	7	5	3	6	5	3	6
9	[illegible]	5	8	7	4	8	7	3	8
10	+ 9	- 6	+11	+ 8	- 5	+ 9	+ 8	- 5	+ 9
11	11	7	13	9	5	11	10	5	11
12	13	8	15	11	6	13	11	6	12
13	15	9	17	13	7	15	13	7	15
14	18	10	20	15	8	17	15	8	17
15	+20	-12	+22	+16	- 8	+19	+16	- 9	+19
16	23	13	25	18	9	21	18	10	21
17	25	15	28	20	10	24	20	11	23
18	28	16	31	23	11	26	23	12	25
19	31	18	34	25	12	28	25	12	28
20	+34	-19	+38	+27	-12	+31	+27	-14	+30
21	[illegible]	21	41	29	13	34	29	14	33
22	41	22	45	32	14	36	32	15	36
23	44	24	48	34	14	39	34	16	39
24	[illegible]	26	52	37	15	42	37	17	41
25	[illegible]	-27	+56	+40	-16	+45	+39	-17	+44

TABLE 3

TOTAL ERROR OF VARIOUS METHODS

t	No Correction	(14)	(33)	(6)
sec	ft	ft	ft	ft
0	0	0	0	0
1	0	0	0	0
2	0	1	0	0
3	2	0	0	0
4	3	0	1	0
5	3	1	1	0
6	5	1	1	0
7	7	1	1	0
8	8	2	0	0
9	11	2	0	0
10	13	2	0	0
11	15	3	1	0
12	17	4	1	0
13	21	4	0	0
14	24	5	0	0
15	26	6	1	0
16	29	7	1	0
17	32	8	1	0
18	36	9	1	0
19	39	10	0	0
20	43	12	2	0
21	46	13	1	0
22	50	15	1	0
23	54	16	2	0
24	58	18	2	0
25	61	20	2	0

COMPARISON OF METHODS

TABLE 4

	No Correction	(14)	(33)	(6)
Computation of corrections	0	1	20	10,000
Computer engineering	0	2	3	500
Value of corrections	0	3	9	10

[redacted]